Continuty equation

We consider the liqid, which moves in a space.
We use cartezian coordinates and consider, that any point of space can be described by three coordinates (x,y,z).
The density of the liquid is a scalar function from coordinates. Let's denote density at the point (x,y,z) at the time moment t as .
Velocity of the liquid in the point of space is the vector-function from coordinates and time. It can be described by its 3 values, , , which are it's projections to the axes at the moment t.

We assume, that functions , , , are defined in a whole space are differentiable there, and their derevative is continious.
Let's consider some parallelipipid in the volume, with sides parallel to the axes.

Let size of sides are $\Delta x, \Delta y, \Delta z$, and coordinates of center of parallelipipid are .
We devide the bottom of parallelipipid into NxN equal squares with sides δx and δy.
Let's observe the system at some moment t_0 for a small period of time Δt
If sides of squares are small enough we may consider, that density and velocity doesent change sufficiently in the square (x,y,x+δx,y+δy) in the small period Δt.
Mass of liquid, which come through the square (x,y,x+δx,y+δy) is
$u_z(x,y,z_0)\rho(x,y,\Delta t\delta x\delta y$
Total mass, which comes through the bottom of parallelipipid is sum of mass comming through all squares

$$\Delta m_z = \sum_{i=1}^{N}\sum_{j=1}^{N} u_z(x_i, y_j, z_0, t_0)\Delta t\delta x\delta y$$
.

When $\delta x, \delta y \to 0$, sum mutates to the integral:

$$\Delta m_z = \Delta t \int_{x_0}^{x_0+\Delta x}\int_{y_0}^{y_0+\Delta y} \rho(x,y,z_0,t_0)u(x,y,z_0,t_0)dxdy$$

Now, we may introduce new function:

$$q_z(x+\frac{\Delta x}{2}, y+\frac{\Delta y}{2}, z, t) = \frac{1}{\Delta x\Delta y}\frac{\Delta m_z}{\Delta t} = \frac{1}{\Delta x\Delta y}\int_{x}^{x+\Delta x}\int_{y}^{y+\Delta y}\rho(x,y,z,t)u(x,y,z,t)dxdz$$

In the middle of the square it has a value, which shows, what mass of liquid goes through the square in z direction per unit of time per unit of square.
One may call it "flow in direction z", or "speed of mass changing in direction z" or whatever his imagination allows him. We will call it "q_z-function".

In the similar way can be defined functions $q_x\left(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}, t\right)$ and $q_y\left(x+\frac{\Delta x}{2}, y, z+\frac{\Delta z}{2}, t\right)$:

$$q_x(x, y+\frac{\Delta y}{2}, z+\frac{\Delta z}{2}, t) = \frac{1}{\Delta y\Delta z}\frac{\Delta m_x}{\Delta t} = \frac{1}{\Delta y\Delta z}\int_{y}^{y+\Delta y}\int_{z}^{z+\Delta z}\rho(x,y,z,t)u(x,y,z,t)dydz$$

$$q_y(x+\frac{1}{\Delta y\Delta z}\frac{\Delta x}{2}, y, z+\frac{\Delta z}{2}, t) = \frac{1}{\Delta x\Delta z}\frac{\Delta m_y}{\Delta t} = \frac{1}{\Delta x\Delta z}\int_{x}^{x+\Delta x}\int_{z}^{z+\Delta z}\rho(x,y,z,t)u(x,y,z,t)dxdz$$

To make formulae shorter, let's denote

$\bar{x} = x_0 + \frac{\Delta x}{2}$, $\bar{y} = y_0 + \frac{\Delta y}{2}$, $\bar{z} = z_0 + \frac{\Delta z}{2}$

Then the mass of liquid, which comes through the bottom side of paralellipipid during the time Δt is $q_z(\bar{x}, \bar{y}, z_0, t_0)\Delta x \Delta y \Delta t$

The mass of liquid, which leaves through the top side of paralellipipid during the time Δt is $q_z(\bar{x}, \bar{y}, z_0 + \Delta z, t_0)\Delta x \Delta y \Delta t$

So, in parallelipipid lefts $\Delta x \Delta y \Delta t\left(q_z(\bar{x}, \bar{y}, z_0, t_0) - q_z(\bar{x}, \bar{y}, z_0 + \Delta z, t_0)\right)$ mass of liquid, which comes through top or bottom side.

For the right-left and front-backward sides all is analogical.
So, total change of mass in the parallelipipid is:

$$\begin{aligned}
&\Delta m = \\
&\Delta y \Delta z \Delta t\left(q_x(x_0, \bar{y}, \bar{z}, t_0) - q_z(x_0 + \Delta x, \bar{y}, \bar{z}, t_0)\right) + \\
&\Delta x \Delta z \Delta t\left(q_y(\bar{x}, y_0, \bar{z}, t_0) - q_z(\bar{x}, y_0 + \Delta y, \bar{z}, t_0)\right) + \\
&\Delta x \Delta y \Delta t\left(q_z(\bar{x}, \bar{y}, z_0, t_0) - q_z(\bar{x}, \bar{y}, z_0 + \Delta z, t_0)\right)
\end{aligned}$$

Let's introduce jet another function: the mean density over the parallelipipid

$$\bar{\rho}\left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}, z + \frac{\Delta z}{2}, t\right) = \bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t) = \int_x^{x+\Delta x}\int_y^{y+\Delta y}\int_z^{z+\Delta z}\rho(x, y, z, t)\,dxdydz$$

Then, change in mass is connected to change of mean density:

$$\Delta m = m(t_0 + \Delta t) - m(t_0) = \bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t_0 + \Delta t)V - \bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t_0)V$$
$$\Delta m = \left(\bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t_0 + \Delta t) - \bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t_0)\right)\Delta x \Delta y \Delta z$$

And we have an equation:

$$\begin{aligned}
&\Delta y \Delta z \Delta t\left(q_x(x_0, \bar{y}, \bar{z}, t_0) - q_z(x_0 + \Delta x, \bar{y}, \bar{z}, t_0)\right) + \\
&\Delta x \Delta z \Delta t\left(q_y(\bar{x}, y_0, \bar{z}, t_0) - q_z(\bar{x}, y_0 + \Delta y, \bar{z}, t_0)\right) + \\
&\Delta x \Delta y \Delta t\left(q_z(\bar{x}, \bar{y}, z_0, t_0) - q_z(\bar{x}, \bar{y}, z_0 + \Delta z, t_0)\right) = \\
&\qquad = \left(\bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t_0 + \Delta t) - \bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t_0)\right)\Delta x \Delta y \Delta z
\end{aligned}$$

If we divide both parts by $\Delta x \Delta y \Delta z \Delta t$ we obtain

$$\frac{q_x(x_0, \bar{y}, \bar{z}, t_0) - q_z(x_0 + \Delta x, \bar{y}, \bar{z}, t_0)}{\Delta x} + \frac{q_y(\bar{x}, y_0, \bar{z}, t_0) - q_z(\bar{x}, y_0 + \Delta y, \bar{z}, t_0)}{\Delta y} + \frac{q_z(\bar{x}, \bar{y}, z_0, t_0) - q_z(\bar{x}, \bar{y}, z_0 + \Delta z, t_0)}{\Delta z} = \frac{\bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t_0 + \Delta t) - \bar{\rho}(\bar{x}, \bar{y}, \bar{z}, t_0)}{\Delta t}$$

And now, if we take limit from that equation when $\Delta x, \Delta y, \Delta z, \Delta t \to 0$
We obtain partial differential equation:

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} = \frac{\partial \rho}{\partial t}$$

The only thing left is to take a look at the difiniton of functions q, when $\Delta x, \Delta y, \Delta z, \Delta t \to 0$

$$q_y(x+\frac{\Delta x}{2}, y, z+\frac{\Delta z}{2}, t) = \frac{1}{\Delta x \Delta z} \int_x^{x+\Delta x} \int_z^{z+\Delta z} \rho(x,y,z,t) u(x,y,z,t) dxdz$$

when $\Delta x, \Delta y, \Delta z, \Delta t$ are small, values of density and velocity do not change sufficiently.
So, in the limit case we may consider $\rho(x,y,z,t) = \rho(x_0, y_0, z_0, t_0)$,

$$q_y(x+\frac{\Delta x}{2}, y, z+\frac{\Delta z}{2}, t) = q_y(x_0, y_0, z_0, t_0)$$

$$u(x,y,z,t) = u(x_0, y_0, z_0, t_0)$$

We have

$$q_y(x_0, y_0, z_0, t_0) = \frac{1}{\Delta x \Delta z} \int_x^{x+\Delta x} \int_z^{z+\Delta z} \rho(x_0, y_0, z_0, t_0) u(x_0, y_0, z_0, t_0) dxdz = \rho(x_0, y_0, z_0, t_0) u_y(x_0, y_0, z_0, t_0) \frac{\int_x^{x+\Delta x} \int_z^{z+\Delta z} dxdz}{\Delta x \Delta z} = \rho(x_0, y_0, z_0, t_0) u_y(x_0, y_0, z_0, t_0)$$

The same is for other components:

$$q_x(x_0, y_0, z_0, t_0) = \rho(x_0, y_0, z_0, t_0) u_x(x_0, y_0, z_0, t_0)$$

$$q_z(x_0, y_0, z_0, t_0) = \rho(x_0, y_0, z_0, t_0) u_z(x_0, y_0, z_0, t_0)$$

So, finally partial differential equation can be rewritten as

$$-\frac{\partial(\rho u_x)}{\partial x} - \frac{\partial(\rho u_y)}{\partial y} - \frac{\partial(\rho u_z)}{\partial z} = \frac{\partial \rho}{\partial t}$$

or $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} = 0$

If we denote $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ then equation may be rewritten in a more compact way:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{u} = 0$$

If the liquid is imcompressible, then it has constant density. In that case equation becomes:

$$\nabla \cdot \mathbf{u} = 0$$